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Renaissance vaults: geometry, nurbs and computational opportunities for reconstruction

Creating an interpretative geometric representation of architectural shapes is crucial to learn the essence of any architecture, to preserve it and to hand it down through the ages to our descendants. Nowadays the generation of an accurate reality-based 3D model from a survey is the first step to set up a geometric model that, before its semantic enrichment, must be geometrically correct.
In this paper, we compare different workflows adopted in modelling a Renaissance vault in Santa Maria delle Grazie complex in Milan starting from a laser scanner survey acquired within a wider survey campaign of the complex. The vault of the Ancient Sacristy, a Renaissance masterpiece by Donato Bramante, has been chosen according to its integrity, despite the bombings suffered by the Complex in 1943. The workflows have been used to optimize a system of comprehension of the building and to evaluate by analogy a similar vault of the complex added with some reflection at the end: the vault of the Refectory that hosts “the Last Supper” painting. This latter was destroyed by bombing and rebuilt last century on the drawing of a direct survey previously occurred. We can consider the vaulted system as a clear example to practice understanding of the geometric rules of architecture first and the vaulted surface modelling then, to be compared through the principles of descriptive geometry in a dialectical relationship between theory and practice between geometry and construction. In this paper, multi-purpose interpretative models try to read and describe the state of the art and possible declinations of the semantics linked to an architectural object unique in its value.

Keywords:
accuracy; vaults;nurbs; parametric; stereometry

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1. STATE OF THE ART

Nowadays, reliable geometry can only be supposed on data from accurate surveys (Balzani, 2020); laser scanner and photogrammetry techniques can record with accuracy the shape of architecture (Tommasi, 2016). Classical studies of geometry regarding vault shapes and typology (Calvo López and Rabasa Díaz 2016) are currently developed on 3D reality-based models (Capone et al. 2015). Moreover, an approach that combines 3D surveyed data and geometric reconstructions (Spallone et al. 2019) can be a valid research method in several sectors: to exploit digital data for structural behaviour employing the 3D model for the finite element process (Tucci et al. 2019); for restoration processes; for historical analysis. The accuracy that the studies of our historical heritage need to improve, the necessary interoperability among different disciplines and BIM environments also oblige us to search for a methodological analysis of modelled shapes to optimize results for the informed models (Brusaporci, 2015). To do so, survey, math (De Carlo, 2019), geometry (Tomasoni, 2015), and construction techniques (Breymann, 2003) are the fundamentals of this research path where the research question is how to classify and manage the 3d model of the specific case study.

2. THE CASE STUDY

This masterpiece of the Milanese Renaissance was built during the last decade of the fifteenth century, when Ludovico il Moro, Duke of Milan, decided to start the renovation project of the Dominican convent of Santa Maria delle Grazie; part of the church previously built by Guiniforte Solari was subjected to some transformations to be used as a mausoleum for the Sforza family. According to Arnaldo Bruschi dating (Bruschi, 1983), the expansion of the convent with a new cloister, and what we call the Ancient Sacristy, is before 1497. This period is characterized by the Ludovico Sforza reign who surrounded himself with artists and cultural figures with a precise idea of the Renaissance in Milan, transforming his court into one of the most beautiful in northern Italy, calling to work, side by side, two of the greatest living artists who had long been at the service of his court such as Bramante and Leonardo. The former entrusted the task of expanding the Basilica with the construction of the Tribuna and the "Ancient Sacristy", which is commonly attributed to him but with no absolute certainty, while the latter was commissioned the Last Supper on the north wall of the Refectory built in the same years (Gattico, 2006). This cultural period, which ended when Ludovico Sforza was chased away by the French in 1499, was the most fruitful from the point of view of the artistic enrichment of the convent. The "Ancient Sacristy" was built in an isolated position concerning the Church, probably due to the limits of the available land (Bascapè e Mezzanotte, 1968) and the aim to leave the maximum prominence to the Tribune of the Basilica. The interior of the Bramante Sacristy is a masterpiece of fascinating legends, geometries, and decorations. Miraculously saved from the bombings of 1943, it has a vault that simulates the blu of a starry night (fig. 1) with multiple golden stars as the vaulted ceiling of

Fig. 1 - The vaulted lunette ceiling of the Ancient Sacristy in a historical picture- 1930, Zuecca U.

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the destroyed Refectory, which had similar shape and decoration; this last one had single lunettes contained by diagonal arches decorated in gold to underline the changes of the curves (Martelli,1980) but 1943 bombing destroyed it leaving us with just two primary documents: a directly measured survey and some picture to describe it.
The vault of the Ancient Sacristy (Zuecca, 1930), and the vaults of the Cloister of the Frogs (Zanzottera, 2015), are some of the few ceilings that remained unscathed from the bombing and from the restorations that then involved the entire complex. It belongs to the family of barrel vaults with lunettes (Lopez, 2016), but two half umbrella vaults complete it on the two minor sides (Docci, 1992) like the Refectory ones.
Only eight years later, Bramante was involved by Michelangelo for the works necessary for the new decoration of the Sistine Chapel realized in 1504 shaped with a similar vault.

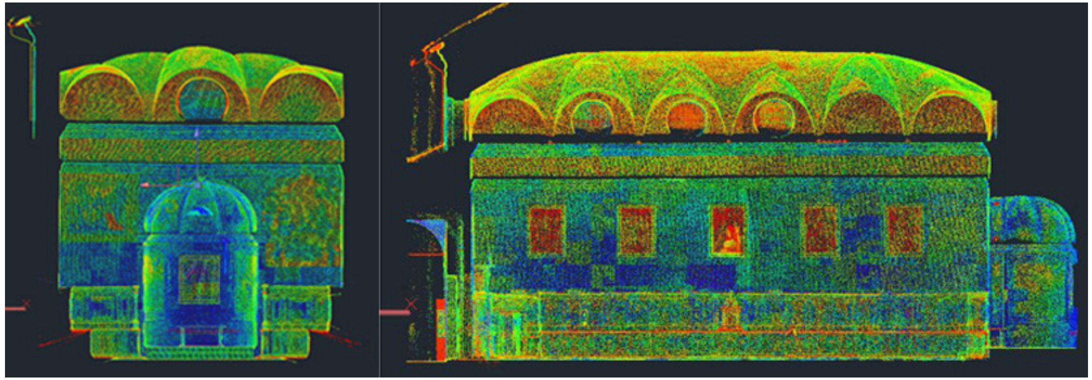

Fig. 2 - Digital survey of the Ancient Sacristy: longitudinal and transversal sections.

3. THE METHOD

The digital reconstruction of the vaults described in the paper follows three workflows to increase the model accuracy progressively and find geometric rules to compare the Refectory reconstruction with the Sacristy: the first workflow after the points cloud acquisition deals with drawing of the curves on the ribbed arches of the vault in order to reconstruct it by simple extrusion from them ; the second workflow relates to an initial approach to the Sacristy vault geometries with NURBS modelling tools considering the survey previously realized; the third workflow relates to the parameterization of the vault using generative tools. The final phase was the comparison of the Sacristy parametrization with the Refectory vault in search of analogy and differences.

4. DIGITAL SURVEY BEFORE GEOMETRICAL RECONSTRUCTION

The whole complex has been surveyed during a two-year campaign with different laser scanner instrumentations that became step by step available. The Cloister of the Frogs was measured with a Leica HDS7000 (phase-shift system with a maximum range of 187m, the maximum scan rate of 1 mio pts/sec). A Leica Geosystems Scan Station P30 (maximum range of 120m, the maximum scan rate of 1 mio pts/sec, 3D accuracy of 3.2mm@50m) was used for the Cloister of the Prior, the new Sacristy, the room of the Candle, and the Old Sacristy. Finally, the interior and exterior survey of the church was performed using a Leica RTC360 (maximum range of 130m, the maximum scan rate of 2 mio pts/sec, 3D accuracy of 6.4mm@50m).
The laser scanner data were processed within the Leica Cyclone 3D Point Cloud Processing Software, using a cloud-to-cloud registration method in a target-less approach. For each survey, a single project with all aligned scans was created. Finally, a rigid roto-translation is applied to have all projects registered in the same coordinate system.
The interior of the Ancient Sacristy was measured employing a Scan Station P30 through eleven scans, while capturing a spherical image per station. Four scans were acquired on each side at about 1.70 m of height; the apse was treated separately with one scan, and two more were added in the centre of the nave at the maximum height possible for the instrument that is 2.50 m. All positioning stratagems were adopted to mitigate the effect of shadows brought to the walls by frames or wooden furniture (fig 2).
The scans were acquired with two different resolutions: 3.1 mm at 10 m for the first nine, the last two, higher, with a resolution of 1.6 mm at 10 m. The photos were captured in HDR full-dome mode with 4096 × 4096 pixels for a single frame. Data acquired inside the Sacristy were from laser scans only, without any needs to capture images as texture through additional instruments, or photogrammetry.
The complete laser scanner model of the Ancient Sacristy has about 120 mio of points. A segmentation of the upper part of the Sacristy regarding the vault geometry has been formed with 56 mio of points.
This segmentation was realized from the lower frame of the vault.
This model was decimated and reduced to 3,5 mio of points, setting a point-to-point distance of 1 cm to manage the point cloud in a modelling software environment efficiently. No other parameters were added to decimate.

5. THE INITIAL APPROACH TO THE VAULT'S GEOMETRY, SURFACES RECONSTRUCTION FROM POINTS

The survey carried out in these years provided the basis for this deeper study where the dataset is necessary to construct the detailed 3D models.
The research question is not only if there is a re-

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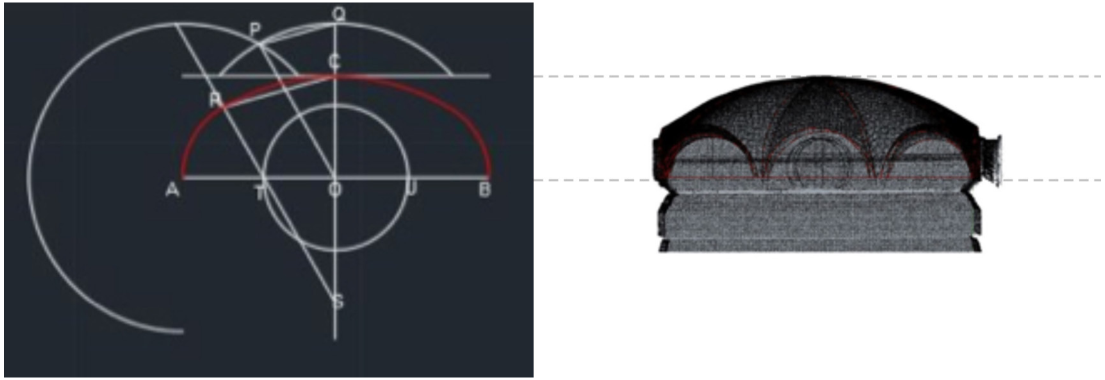

Fig. 3 - The study of the main generating lines of the Vault of the Sacristy; on the left the main curve, derived from a 3-circle arch with three circles on the right the point cloud.

Fig. 4 - The Sacristy's ceiling of the Sacristy vault derived from geometrical reconstruction across sections of the point cloud extruded along ropes.

sulting curve corresponding to a precise geometry shape of the Sacristy Vault but how to describe this and if this exact shape could be considered the correct one for the Refectory in its original shape also.
In this phase, we started focusing on geometric analysis of the Sacristy vault, considering the possibility of modelling them using NURBS surfaces derived from point clouds adopting a twofold procedure:

- the first one was based on modelling starting from wireframe, using closed exterior edges that act as controllers of the surfaces and are based on euclidean geometry. The points are concatenated to describe the edges, considering possible geometric relationships in terms of dimension. These points interpolation can be considered as the main matrix that informed the realization of the vault; the ribs drew with a system of discharges the main warps of the intersection between the vaulted planes (Tomasoni, 2015);
- for the second procedure we used control points selected in the NURBS operating as precise points activated in the digital point cloud that can be modified precisely because they are recognized by Cartesian coordinates in a specific software environment.

In the first case the logical path started from finding the shape of the arches of the main vault in consideration of its key points and major chord; surfaces were derived as subsequently as extrusions, thearches of the lunettes extruded and intersected with the main vault along their rope finally, we focused on the arrangement of the terminal parts of the umbrella vaults. The generatrixes were considered as the most coherent figures with respect to the sections derived from the decimated cloud: three-circles arch for the major curve (fig.3-4); wider arch on the lunette vaults; composition of curves for the umbrella vaults. It has been a research brought through drawings among the edges of the curves.
However, the comparison between the point cloud and the supposed three-arches (with no shift of axes for regular construction) construction give rise to a visible deviation up to 20 cm (fig.5) in the intersection between the lunettes and the main arch. In the second procedure, the complete model generated with Rhinoceros 6 was obtained considering NURBS surfaces con-

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trolled by a closed exterior edge derived from the point cloud, and internal section curves to generate the shapes.
However, despite the precision of the NURBS model, it revealed several deviations if not errors in the hypothesized structure. The deviation has been automatically calculated with Geomagic software also.
Given the complexity of the vault, the second modelling workflow used NURBS algorithms to generate a model capable of maintaining the morphological uniqueness intercepted by the 3D survey instrument (P30 Laser scanner) and the relative resolution. The procedure uses in combination:

- the determination of the geometric primitives capable of representing the internal / external edges of the main elements that make up the object under consideration made in an automatic procedure by extracting the edges of the general geometry of each element from the cloud;
- the generation of NURBS surfaces with a 32x32 UV subdivision through direct interpolation of the points that make up the scans (fig.6).

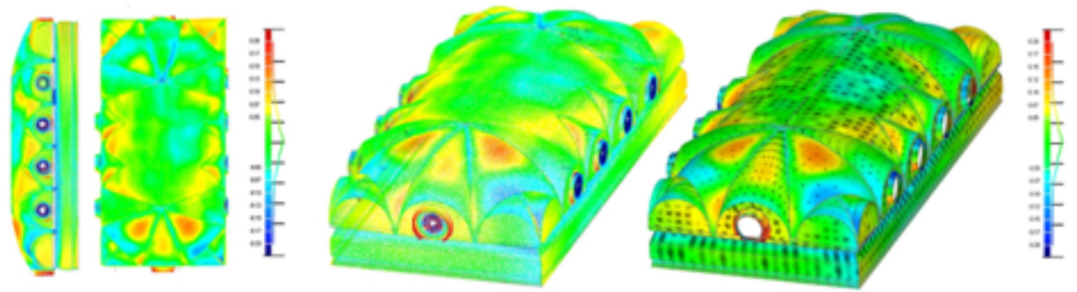

Fig. 5 - Most of the deviations between the ideal and the real surface are within the + -5cm range (as seen from the histogram). This value corresponds to the resolution of the cloud points after sampling. The trend of the shape differs up to 20 cm in the zone of vault's ribs (where there is the maximum curvature). This can be an inaccuracy of the modelling. n the lower part of the deviations map there is an area with a large deviation of -15 cm, presumably due to the irregularity of the real curve. Axonometries: on the left, the deviations between ideal and real surface mapped on the cloud; on the right the deviations related to the mathematical model.

Fig. 6 - The workflow sequence of the second workflow.

In particular, the process was based on the following steps:

1. Importing the point cloud into the NURBS modelling software;
2. Determination of geometric primitives by manual 3D drawing from a point cloud, determining a closed geometric profile and a correct interpretation of each element to be returned by the modeler;
3. Generation of the individual construction elements that make up the vault through NURBS interpolation. The geometric primitives necessary for the automatic generation of the model are:
   (i) the main three-dimensional edges that intercept the general geometry of each individual element (see the previous point) and (ii) the points that make up the point cloud (145297 points);
4. Analysis of the degree of accuracy of the created scan-to-BIM model. For the specific case of the vault, the deviation obtained is equal to 1.5 mm, defining the accuracy of the model created and its level of detail.

6. GENERATIVE FORM FINDING ANALYSIS

In this workflow, we used a generative tool to verify different hypotheses on the geometric constructs of the vault; this tool has been set to find a parameters matrix to minimize the deviation from points cloud, then we developed an algorithm to generate variations of the vault consistent with the

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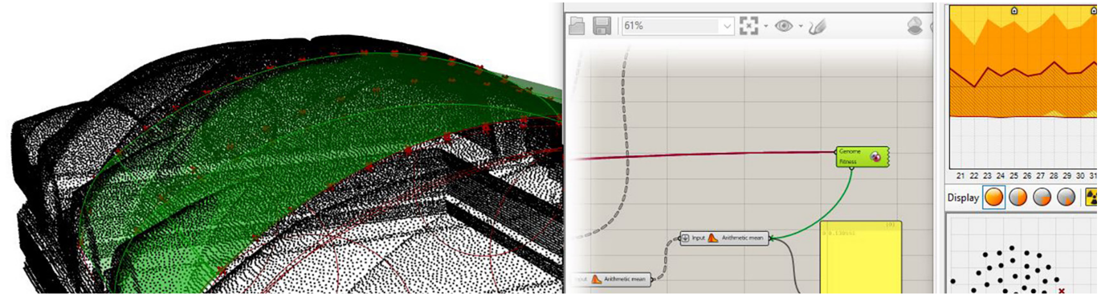
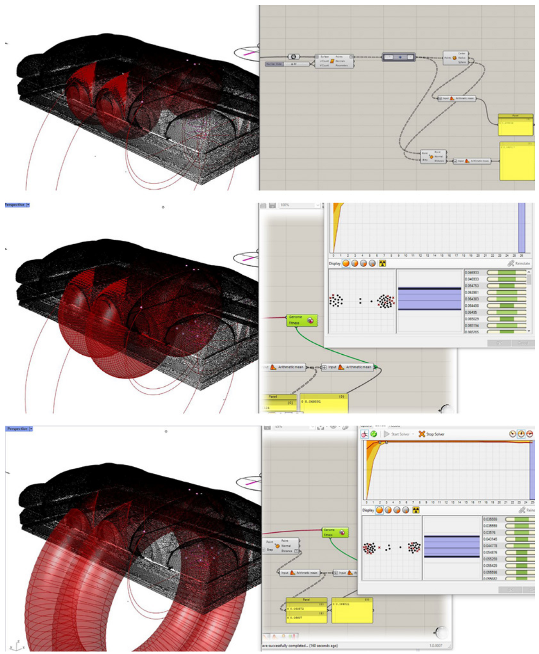


Fig. 7 - Generative research of the three-centre arch consistent with the original model sections.

original one. This second algorithm was used to identify possible differences between the Sacristy and the Refectory. Using the Rhinoceros 7.3 parametric plugin Grasshopper 1.0, the Sacristy vault has been shaped following two phases: 1) primitive geometries checking and validation through the generative tool; 2) identification of geometric constructs related to the analysis. The first investigation started from a reconstruction of the central vault's primary geometries with possible primitive surfaces such as cylinders, toroids, spheres, ovoids, considering the NURBS surfaces reconstruction and an iterative comparison with the point cloud. For each of the surfaces analysed below, the related symmetry planes have been identified through the generative tool of Grasshopper *Galapagos*. This operation implies identifying a bounding box with the minimum volume capable of containing the original model's geometries; in this way, the identified axes of symmetry are not chosen a priori but obtained by analysing the model itself. Using the symmetry planes, we generated several different primitive geometries (cylinder, sphere, torus, ovoid) aligned to the reference geometries of the vaults. For each of these classes of shapes, we used Galapagos to inquiry all possible radii, rotation and movement parameters combinations in search of a reliable shape with the minimum drift from the point cloud. Cone or cylinder are not considered valid shapes due to the curvature of the lunettes' cross-section. The geometries analysis has been carried out separately, first for the central vault and then the umbrella vaults (fig.7).

Fig. 8 - Hypothesis of the generating shapes, from top to bottom: i) spheres; ii) single toroids; iii) toroids by pairing lunettes.



First phase: primitive geometries checking, and validation through the generative tool

Main vault: the first step to reconstruct the vault geometries have been finding three sections delineating the original model's two bays. For each section, a simulation has been started using Galapagos to test all possible positions of the three centres related to the arch segments to identify the combination of parameters that would result in the minimum distance from the original sections without generating cusps. The most plausible hypothesis among the proposed alternatives implies lateral arch segments that subtend an angle of about 60° between the ends and the centre. This geometry is consistent with the isometric oval hypothesis advanced in the first reconstruction; the arch determined in this way has an average distance of 0.08 m with a minimum spacing of 0.03 m. Reconstructing the circumferences relative to the lunettes' middle arches and considering the surfaces' double curvature, the analysis was oriented on the possibility of spherical segments or toroidal surfaces.Therefore, four hypotheses were investigated: i) spherical segments; ii) single toroidal segments; iii) opposing lunettes belonging in pairs to the same toroid (fig.8); iv) ovoid segments.
Hypothesis (i) the average radius of the approximation spheres is 2.49 m, the drift from the original surfaces with these parameters is 0.048 m.
Hypothesis (ii) the average radius of the equatorial circumferences of the four toroids is 3.97 m, the radius of the section circle that best approximates the lunettes is 1.88 m, the drift from the original surfaces with these parameters is 0.047 m.
Hypothesis (iii) (fig.9) the mean radius of the equatorial circumferences of the two toroids is 10.30 m, the radius of the section circle that best approximates the lunettes is 1.85 m, the drift from the original surfaces with these parameters is 0.035 m.
Hypothesis (iv), the resulting ovoid was so longitudinally stretched to resemble a capsule; for this reason, it is not considered reliable.
Hence, hypothesis (iii) was the most reliable. The centre of the toroid of each pair was identified, and the average position between the two points was

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estimated to get the ideal position of the centre of the geometry concerning the impost of the vault and the axis of symmetry of it. The outer arches circumferences' radius of the lateral lunettes is an average of 1.66 m.

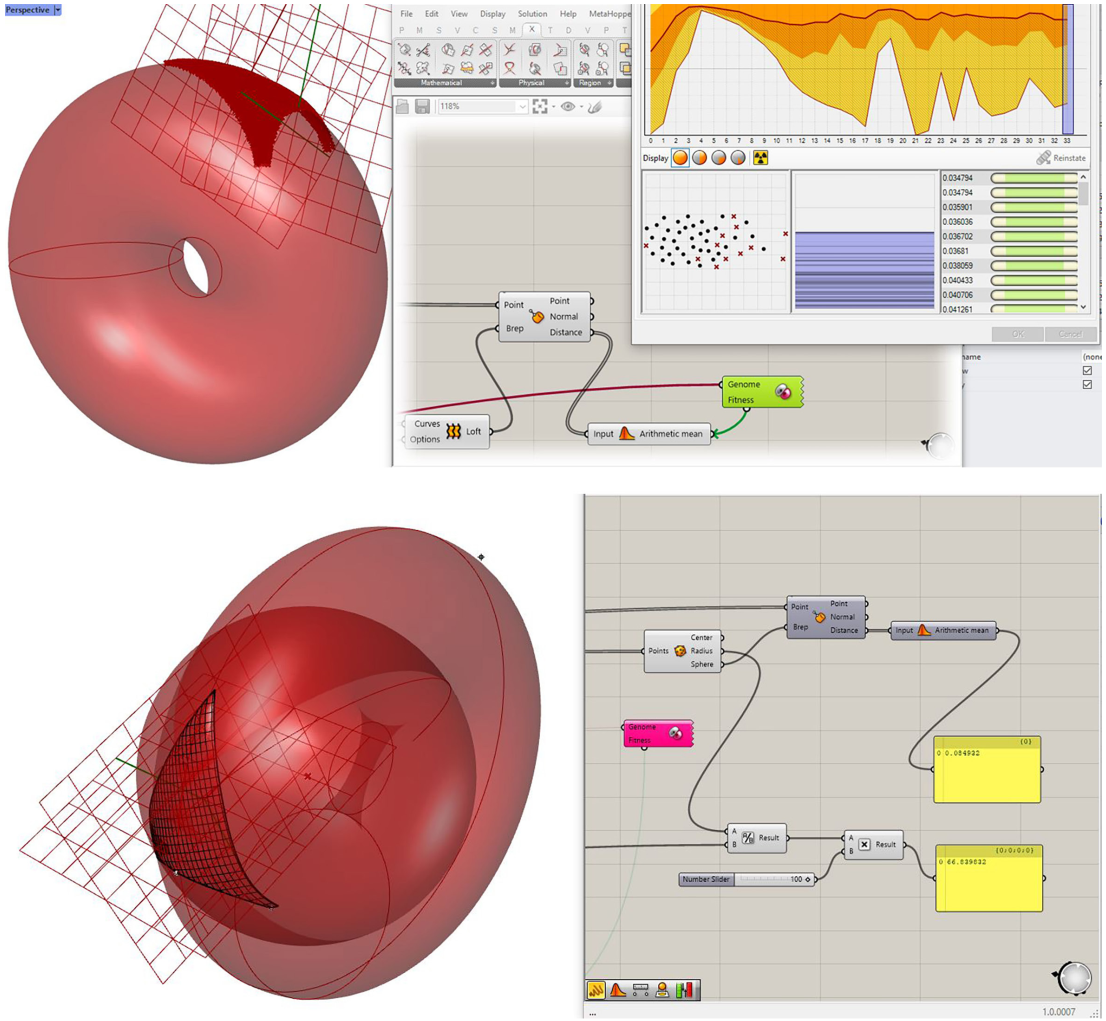
Umbrella vault analysis

In search for the possible geometry to which the diagonal sails belong, we checked two possibilities: i) portion of a sphere; ii) portion of a toroid. Assuming the first hypothesis we then identified the sphere that most accurately approximates the sail's geometry. We verified the average distance of the sail points to the sphere's surface; this distance was found to be 0.08 m with a radius of 4.85 m.
Assuming the second hypothesis, we proceeded by identifying the circumference to which the median arch of the sail belongs, and we used the Galapagos generative tool (fig.7-8) to identify the section circumferences that would result in the minimum average distance between the points of the sail and the surface of the toroid; this distance turns out to be 0.03 m with a radius of 3.16 m. It should also be considered that the ratio of the hypothetical approximation sphere's radius and the original medial arch of the sail's radius (fig. 10) is 0.67. Hence the toroid hypothesis is the most reliable, considering the narrow drift of the toroid relates to the sail's surface and the inconsistency between the radius of the sphere and the radius of the medial arch.

Second phase: identification of geometric constructs related to the analysis

Once the average radii and the curves' centers' average positions were identified, we proceeded to calculate the proportions and the possible module that were common to the construction of the identified generating geometries. Observation and comparison led us to consider the distance between the consecutive lunettes along the Sacristy longitudinal axis as a measure of the module M. We verified that M is equal to 1/36 of the Sacristy shorter side.

Fig. 9 - Generative search for the toroid that best approximates the diagonal sail (up); Comparison between the approximation sphere and the toroid based on the median arch (down).



The three-centre arch of the vault (fig 10) is formed by lateral arch segments which centres are located at 1/4 and 3/4 of the shorter side of the Sacristy; their radii lengths is 1/4 of the vault side. The central arch segment has radius SP equal to √(2&(2QP)^2-(AB/4)^2) ; the center is

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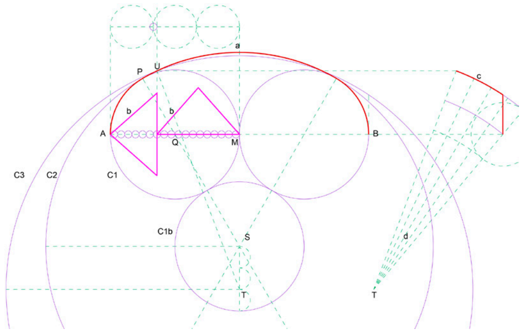

Fig. 10 - Construction of the vault's section arch with three centers and lateral lunettes: A-B: shorter side of the Sacristy; P: point of intersection between C2 and C1a; Q: center of the circumference C1 with radius QP; S: centre of the circumference C2 with radius SP; T: centre of the circumference C3 with radius TU; U: point of intersection between C2 and C3; M: base modulus equal to 1/18th of the diameter of C1, this is also the distance between the adjoining lunettes; a) section of the vault; b) plan of the vault lunettes; c) extract of the toroidal lunettes section; d) directrices of the toroid section planes.

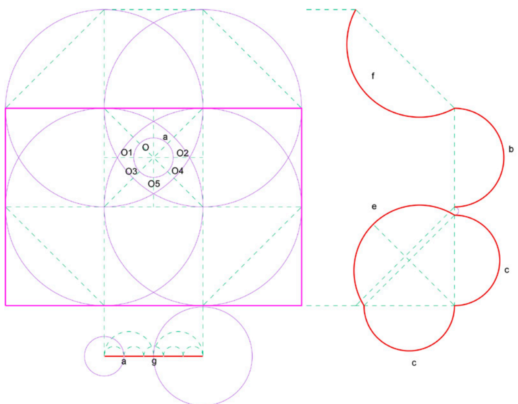

Fig. 11 - Distance from the umbrella vault's center "O" and the of the lunettes toroids' centers (O1-5), equal to 1/15 of the vault base square's side; b: the arch of the lunettes on the longitudinal-transverse axes; c: the arch of the corner lunettes; e: the arc of the diagonal sails next to the corner lunettes; f: the arch of the diagonal sail near the longitudinal and transverse lunettes; g: height at the arch key, equal to 1/6 of the base square side..

identifiable as the intersection "S" of the 60° directrices that originate in the centers Q of the two lateral arch segments.

The side lunettes (fig. 11) have a triangular shape in plan (fig.10); this triangle has a height of 6.5 M and a base of 11.5 M. It can be observed that the sum of the base length of the triangle and its height represents exactly the diameter 2QP of the circle that subtends the side arch segments of the vault itself. The base triangle height also establishes the point of intersection "U" between the equatorial circumferences, generators of the lunettes, and the three-center arch vault.

The center "T" of the generating circumferences of the side lunettes' toroids is located at a distance equal to 2QP/3 below the point S, thus defining the radius TU; the diameter of the toroid section turns out to be equal to 9 M.

The umbrella vault is constructed based on a square of side AB (shorter side of the Sacristy). The octagon that determines the shape of the lunettes in the plan is drawn starting from the square sides' tripartition. In particular, the area covered by the vault is drawn on a rectangle whose sides are in the ratio of 2/3. The equatorial circumferences of the longitudinal and transverse lunettes' toroids have radius OB, half of the square diagonal; the centres of these circumferences are positioned in the plan at O1-2-3, respectively, points of intersection between a circumference of radius 1/15 of AB (fig. 11) (with centre coincident with the centroid of the base square) and the longitudinal and transversal axis of the square. These points are located at a distance vertically equal to OB from the central vault's key (fig. 12, O1-2).

The terminal arches of the lunettes on the umbrella vault cross (fig. 10, b) have a diameter equal to AB/3; the terminal arches of the corner lunettes (fig. 10, c) have a diameter equal to AB/3-M; the terminal arch of the diagonal sails (fig. 11) that connects to the corner lunettes has a height in key equal to AB/6 (fig. 10). The centres of the diagonal sails' circumferences are at a distance in plan from the centre O equal to 3 M and a distance in vertical from O equal to OB*7/8 (fig. 12); this measure also constitutes the radius of the diago-

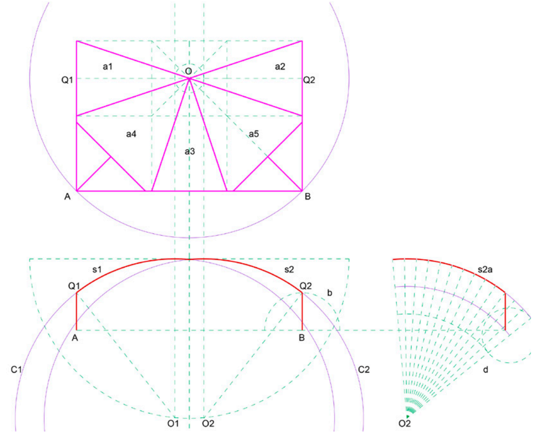

Fig. 12 - The longitudinal and transversal lunettes sections construction for the umbrella vault: A-B: smaller side of the Sacristy; O1: center of the circumference C1 with radius OB; O2: center of the circumference C2 with radius OB; Q1,2: the intersection between the construction plane of the lunette arch and the generating circumference of the toroid; a1-3: the plan of the lunettes along the transverse and longitudinal axis of the Sacristy; s1,2: sections of the lunettes a1 and a2, respectively; s2a: extract of section s2; d: directrices of the section planes of the lunette toroid.

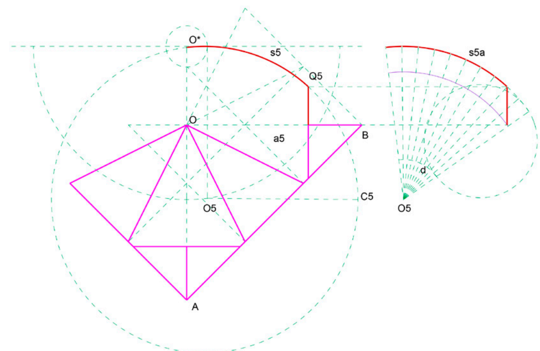

Fig. 13 - Diagonal sail section construction. A-B: side of the vault in the plan; O5: center of the circumference C5 with radius OB*7/8; Q5: the intersection of the construction plane of the arch of the sail and the generating circumference of the toroid; a5: the plan of the diagonal sails; s5: section of the sail a5; s5a: extract of the section s3; d: directrices of the section planes of the toroid related to the sail.

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nal sails' toroid equatorial circumference. The radius of the toroids of the diagonal sails' sectional circumference is equal to 8 M.

7. THE REFECTORY'S VAULT COMPARISON

To understand the geometry of the vault of the Refectory, close to the construction of the Sacristy but in a location surely more constrained, the drawing of the survey of the Superintendence of monuments was taken as a reference for the construction of the model (Soprintendenza ai beni architettonici, 1879). According to historical documents, the two vaults appear to have been built within a few months, apparently by the same hand or, in any case, by the same craftsmen. However, the vault of the Refectory immediately appeared to be more imposing than that of the Sacristy, so much so that over time it required some buttresses in the wall facing outwards. In any case, the Soprintendenza's document would seem to be more accurate than the reconstruction made by Pica (realized before 1943), and although it is not possible to objectively detect differences due to the accuracy derived from the state of origin of the documents (cloud of points/ direct survey of the two vaults) it does give rise to substantial observations.
We translated the geometrical system identified in the Sacresty into an algorithm that can adapt the surfaces based on room sizes and the number of bays. Applying the constructs identified for the Sacristy to the Refectory, it is immediately evident that the arch of the central vault would be 0.53 m lower than the actual arch. The arch of the Refectory's vault is not based on a radius equal to half the short side; the lateral arches have radii 2.78 m, and the central arch has a 5.46 m.
The base triangles of the lateral lunettes have a height-base ratio equal to 2/3, while in the Sacristy, the base and height summed up corresponded to half of the vault's short side; this makes the base triangles of the Refectory's lunettes more acute (with an angle at the top equal to 74.1°) compared to the Sacristy (83°). Furthermore, the lunettes' terminal arch does not have a diameter equal to 1/3 of the short side as in the Sacristy, but a ratio equal to 1/2.67.
Analyzing the proportions of the geometric construction elements of the Refectory we found out that the golden section is the regulatory value (fig. 15). In particular, the triangle at the base of the lateral lunettes is the doubling of a Diophantine triangle (right triangle with sides in the ratio 1:Φ:√Φ) with the major cathetus equal to O1-E. The diameter of the umbrella vault's lunettes arches is obtained directly from the progressive golden section of the side AB (short side of the umbrella vault area); the diameter DB of the corner lunettes' arches is equal to AD/Φ; the diameter AC of the umbrella vault's transverse and longitudinal lunettes terminal arches is equal to AD-GF, where GF is the side of a square obtained by the progression of the golden section.

7. CONCLUSIONS

The in-depth analyses carried out on the vault of the Old Sacristy of St. Mary's have demonstrated the accuracy possible in the study of the forms that have survived to the present day. Digital sur-

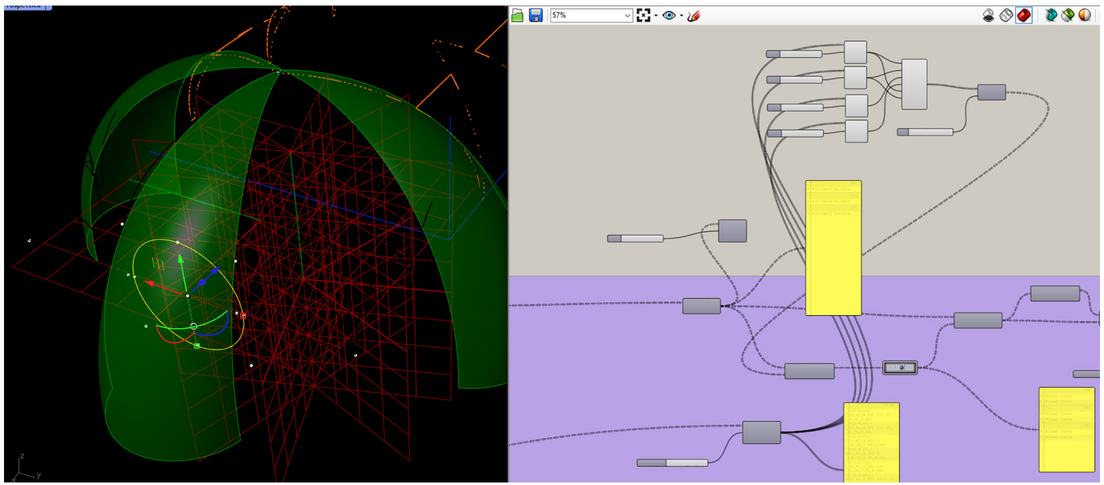

Fig. 14 - comparison of the constructs identified in the Sacristy and the geometries of the Refectory.

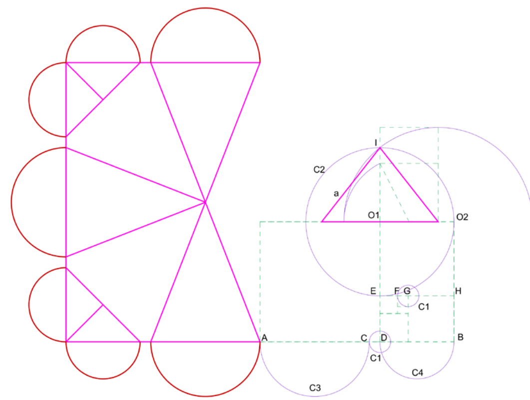

Fig. 15 - Umbrella vault plan and geometric constructs. C1: circumference of radius GF; C2: circumference of radius O1-E; C3: projection of transverse and longitudinal lunettes' terminal arch; C4: projection of corner lunettes' terminal arch; a: plan of the base triangle of side lunettes of the central vault.

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veying, modelling from NURBS, and parametric modelling proved to be valid systems for verifying clear geometry against the current form. Further developments of this process will be to import the NURBS model into the BIM platform to proceed with the parameterization process of the vault and associate specific structural information to the geometries. In case of inspections or survey on the extrados, it is possible to adjust the thickness of the vault to the real value, avoiding remodelling the element.
This entire process applied with this case study increased the degree of accuracy in the geometric reconstruction of the vaults progressively, allowing us to consider changes over time compared to the original construction and deviations from the original structure. The deviations found between the modelling and the actual state have opened up several of questions not only about the state of maintenance of the spaces but also about the reconstruction of part of it (in the case of the refectory vault). The difference in deflection was interpreted as the effect of construction settling over time with relative lowering of the arch in compression.
Anyhow, creating a parametric model of the Sacristy vault has demonstrated the possibility of using it as a benchmark for other vaulted rooms of the same period, opening up the possibility of further comparisons and analyses being carried out in the same way on other monuments.

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